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A revisit to MPC of discrete-time nonlinear systems

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Abstract— **In this paper, model predictive control (MPC) of discrete-time nonlinear systems with guaranteed nominal stability is revisited. In general the optimal cost function of the optimization problem might be discontinuous in the state of the systems, though it is not valuable to be chosen as a candidate Lyapunov function. Compared with the existing results, in this paper the optimal cost function is only used to show the convergence of the system trajectory to its equilibrium. Instead stability is proven in terms of a candidate Lyapunov function which is locally twice continuously differentiable in a vicinity of the equilibrium. Furthermore, asymptotic stability is achieved by the stability of the considered systems together with the convergence of the system trajectory to its equilibrium. In the end, locally robustly asymptotic stability of model predictive control is proven based on the locally continuous Lyapunov funciton. That is, locally inherent robustness of MPC of nonlinear systems with respect to input constraints, state constraints and terminal constraints is proven.**

I. INTRODUCTION

Model predictive control (MPC), also referred to as receding horizon control or moving horizon control, is one of the most widely used approach for advanced control of complex and chemical systems due to the ability to systematically deal with multivariable nonlinear dynamics and constraints [1–4]. At each sampling instant, a control sequence is obtained by solving a finite horizon open-loop optimal control problem which uses the current state of the plant as the initial state. Then only the first control action in this sequence is applied to the plant.

In order to guarantee the stability of nonlinear model predictive control, an intuitive way is to choose an infinite prediction horizon [5]. In engineering practice a long enough prediction horizon is adopted to replace the infinite prediction horizon. However the computational burden is still too heavy to implement. Another intuitive way to guarantee stability is to add a terminal constraint of $x(t+T) = 0$ in the optimization problem which tries to force the terminal state to its equilibrium [6]. Since the optimization problem is in general nonconvex and nonlinear, the extra equality will also increase the computational burden dramatically. Compared with [6], a terminal inequality of $x(t+T) \in \Omega$ is added in the optimization problem, where Ω is an ellipsoid centered at the equilibrium [7]. The optimization problem is solved online to calculate the control sequence while the system state is outside the terminal set of Ω, and a linear control law calculated offline is applied directly while the system state is in Ω. In principle, stability of this model predictive control scheme is guaranteed by the chosen linear control law. Both the terminal control law and terminal penalty are added in the optimization problem to guarantee the stability of systems in $[1, 8, 9]$, where the optimization problem is solved online and the optimal cost function is used as the candidate Lyapunov function. Both recursive feasibility and asymptotic stability are guaranteed if the optimization problem is feasible at the initial time instant. Since only feasible solution is needed to guarantee stability, the scheme is one of the most important MPC schemes with guaranteed nominal stability.

The optimal cost function is first employed as a Lyapunov function for establishing stability of model predictive control of constrained time-varying nonlinear discrete-times systems in [10]. Thereafter, the optimal cost function was almost universally employed as a natural Lyapunov function for stability analysis of model predictive control [1]. The main difficult to choose the optimal cost function as a candidate Lyapunov function in the analysis of the stability is that it is hard to show the continuity of it in the state x. Furthermore, the optimal cost function might be non-continuous on the system state $[3, 11, 12]$ in some cases.

In this paper, stability of finite horizon model predictive control of discrete-time nonlinear systems is revisited. Firstly the existence of the terminal set and the terminal penalty is proven. Secondly, the properties of the optimal cost function of finite horizon model predictive control are discussed: continuous at the equilibrium, semi-positive definite and monotonically decreasing along its trajectory. Thirdly, convergence of the system trajectory to the equilibrium is shown by the monotonically decreasing property of the optimal cost function. Since stability only reflects the local properties of system dynamics, the terminal penalty is chosen as the candidate Lyapunov function in the terminal set. Asymptotically stable is proven since the system is stable and convergent. The system is robustly asymptotically stable (aka inherent robust) in the terminal set.

The organization of this paper is as follows. An introduction followed with notations, basic definitions and preliminary results is given in Section I. The general problem set-up, including the existing of the terminal set and terminal penalty, is given in Section II. The properties of the optimal cost function as well as the recursive feasibility of the optimization problem are discussed in Section III. The convergence, asymptotic stability and locally robustly asymptotic stability of finite horizon MPC of nonlinear systems, are discussed in Section IV. Some concluding remarks

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are provided in Sectin V.

A. Notations and Basic Definitions

Let $\mathbb{R}$ denote the field of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}$ the field of non-negative integers, $\mathbb{Z}_+$ the field of positive integers, $k+i|k$ the predicted value at the time instant $k+i$ starting from the time instant k. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the 2-norm and $\|v\|_Q = \sqrt{v^TQv}$ with $Q \in \mathbb{R}^{n\times n}$ and $Q > 0$. The matrix I denotes the identity matrix with compatible dimension. Let $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of the eigenvalues of matrix M. A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ function, if it is strictly increasing, $\alpha(s) > 0$ for all $s > 0$, and $\alpha(0) = 0$. A continuous function α is said to belong to class $\mathcal{K}_\infty$ function, if it is a $\mathcal{K}$ function, and $\alpha(s) \to \infty$ as $s \to \infty$.

B. Preliminary results

Before proceeding it is necessary to introduce the definitions of stability and asymptotic stability, and a sufficient condition for Lyapunov stability of discrete-time nonlinear dynamical systems.

Consider discrete-time nonlinear systems:

$$x_{k+1} = g(x_k), \qquad x_0 = x(0) \tag{1}$$

where $g(x)$ is piecewise continuous, and $g(0) = 0$.

Definition 1: [13, Definition 13.1]
(i) The zero solution $x_k \equiv 0$ to (1) is stable if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_0\| \le \delta$, then $\|x_k\| \le \epsilon$ for all $k \in \mathbb{Z}_+$.
(ii) The zero solution $x_k \equiv 0$ to (1) is asymptotically stable if it is stable and there exists $\delta > 0$ such that $\|x_0\| \le \delta$, then $\lim_{k\to\infty} x_k = 0$.

Lemma 1: [13, Theorem 13.2]
The zero solution $x_k \equiv 0$ to (1) is stable, if there exists a continuous function $S(x) : \mathcal{H} \to \mathbb{R}$ such that
(i) $S(0) = 0$, and $S(x) > 0$ for all $x \neq 0$,
(ii) along the system trajectory, $S(x)$ satisfies

$$S(x_{k+1}) - S(x_k) \le 0, \quad \forall x \in \mathcal{H}, \tag{2}$$

where the set $\mathcal{H}$ is a compact set.

II. Problem Setup

Consider discrete-time nonlinear systems:

$$x_{k+1} = f(x_k, u_k), \tag{3}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ are the system state and control input at time instant k, respectively. The constraints of the system state and the control input are as follows:

$$x_k \in \mathcal{X},\ k \ge 0\,, \tag{4a}$$
$$u_k \in \mathcal{U},\ k \ge 0\,, \tag{4b}$$

where $\mathcal{X}$ is the admissible set of the systems state, and $\mathcal{U}$ is the admissible set of control input.

In this paper, we assume that all states x_k are measurable instantaneously and there is neither external disturbances nor model perturbations at all.

The following assumptions are required for system (3):

Assumption 1: $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_x}$ is twice continuous differentiable, $f(0, 0) = 0$. That is, $(0, 0) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ is an equilibrium of system (3).

Assumption 2: $\mathcal{U} \subset \mathbb{R}^{n_u}$ is compact, $\mathcal{X} \subset \mathbb{R}^{n_x}$ is connected, and the point $(0, 0)$ lies in the interior of the set $\mathcal{X} \times \mathcal{U}$.
The assumption $f(0, 0) = 0$ is not restrictive, since one can shift the origin of the system to (x_s, u_s) by the transformation of $x' = x - x_s$, $u' = u - u_s$, if $f(x_s, u_s) = 0$.
At the time instant k, define a sequence of the control input

$$U_k := \{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}\}. \tag{5}$$

The open-loop optimization problem of the finite horizon model predictive control at the time instant k is formulated as follows:

Problem 1:

$$\underset{U_k}{\text{minimize}} \quad J(x_k, U_k) \tag{6a}$$

subject to:

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}), \quad x_{k|k} = x_k, \tag{6b}$$
$$x_{k+i|k} \in \mathcal{X}, \qquad i \in \mathbb{Z}_{[1,N-1]}, \tag{6c}$$
$$u_{k+i|k} \in \mathcal{U}, \qquad i \in \mathbb{Z}_{[0,N-1]}, \tag{6d}$$
$$x_{k+N|k} \in \Omega, \tag{6e}$$

where

$$J(x_k, U_k) = \sum_{i=0}^{N-1} \|x_{k+i|k}\|_Q^2 + \|u_{k+i|k}\|_R^2 + \|x_{k+N|k}\|_P^2, \tag{7}$$

and Q and R are positively definite matrices. The terminal set Ω and the terminal penalty $\|x_{k+N|k}\|_P^2$ will be introduced in the future.

The control objective of the finite horizon MPC is to achieve a finite horizon control sequence by solving Problem 1 online such that system (3) is stable, the performance (7) is minimized and the constraints (4) are satisfied.

Although a finite horizon control sequence will be achieved by solving the optimization problem, only the first control action in the sequence will be applied to the considered systems. At the next time instant, the whole process will be run again. For inconvenience, we will first define the feasibility of the optimization problem.

Definition 2: Suppose there exists a control sequence U_k such that

(i) for all $i \in \mathbb{Z}_{[0,N-1]}$, the constraints (4) are satisfied;
(ii) the terminal constraint $x_{k+N|k} \in \Omega$ is satisfied;
(iii) the cost function (7) is bounded, i.e.

$$\sum_{i=0}^{N-1} \|x_{k+i|k}\|_Q^2 + \|u_{k+i|k}\|_R^2 + \|x_{k+N|k}\|_P^2 < \infty.$$

Thus, Problem 1 is feasible, and the control sequence U_k is a feasible solution to Problem 1.

Suppose that Problem 1 has an optimal solution U_k^* at time instant k,

$$U_k^* = \{u_{k|k}^*, u_{k+1|k}^*, \ldots, u_{k+N-1|k}^*\}, \tag{8}$$

and the corresponding optimal trajectory are denoted as X_k^*,

$$X_k^* := \{x_{k+1|k}^*, x_{k+2|k}^*, \ldots, x_{k+N|k}^*\}.$$

Thus, the actual control at the time instant k is

$$u_k := u_{k|k}^*.$$

Consider the Jacobian linearization of the nonlinear system (3) at the equilibrium $(0,0)$,

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k, \tag{9}$$

where $A := \frac{\partial f}{\partial x}|_{(0,0)}$ and $B := \frac{\partial f}{\partial u}|_{(0,0)}$.

Assumption 3: System (9) is stabilizable.

That is, there exists a linear state feedback control law $u = Kx$ such that the controlled system $A_k := A + BK$ is asymptotic stability. For such a given K, the following lemma can be proven.

Lemma 2: Suppose that Assumptions 1-3 are satisfied, respectively. Then, (i) Lyapunov equation

$$\kappa^2 A_k^T P A_k - P = -(Q + K^T R K) \tag{10}$$

admits a uniquely positively definite solution P, where $\kappa \in \left(1, \frac{1}{\tau_{\max}(A_k)}\right)$ and $\tau_{\max}(A_k)$ is the maximum magnitude of the eigenvalues of matrix A_k.
(ii) There exists $\alpha > 0$ which specifies an ellipsoid

$$\Omega := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha\right\},$$

such that

- $\Omega \subseteq \mathcal{X}$,
- $Kx \in \mathcal{U}$ for all $x \in \Omega$,
- Ω is positively invariant for the nonlinear system (3) with the linear control law $u = Kx$,
- for any $x \in \Omega$, the cost function (7) for the nonlinear system (3) starting from $x_0 = x$, with the linear control law $u = Kx$, is bounded from above as follows

$$\sum_{k=0}^{\infty} \|x_k\|_Q^2 + \|u_k\|_R^2 \le x^T P x. \tag{11}$$

Proof: (i) For all $\kappa \in \left(1, \frac{1}{\tau_{\max}(A_k)}\right)$, all the eigenvalues of κA_k lie in the unit circle since A_K is Hurwitz. Thus, Lyapunov function (10) has a uniquely positively definite solution as $Q + K^T R K$ is positive.
(ii) Since the equilibrium $(0,0) \in \mathcal{X} \times \mathcal{U}$, there is a positive constant α_0 such that in the set

$$\Omega_0 := \{x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha_0\},$$

the state constraints (4a) and control constraints (4b) are satisfied.

The time difference of $x^T P x$ along the trajectory of $x_{k+1} = f(x_k, Kx_k)$ is

$$\begin{aligned}
&x_{k+1}^T P x_{k+1} - x_k^T P x_k \\
=&(A_k x_k + \phi_k)^T P (A_k x_k + \phi_k) - x_k^T P x_k \\
=&\kappa^2 x_k^T A_k^T P A_k x_k - x_k^T P x_k + (1-\kappa^2) x_k^T A_k^T P A_k x_k \\
&+ x_k^T A_k^T P \phi_k + \phi_k^T P A_k x_k + \phi_k^T P \phi_k,
\end{aligned} \tag{12}$$

with $\phi_k := f(x_k, Kx_k) - A_k x_k$. Furthermore, in terms of Taylor's theorem,

$$\phi_k = \frac{1}{2} x_k^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k^2} x_k + \frac{1}{2} x_k^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k \partial u_k} K x_k + \frac{1}{2} x_k^T K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k \partial x_k} x_k + \frac{1}{2} x_k^T K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k^2} K x_k$$

for some ξ_k between 0 and x_k.

Denote

$$C_M(\xi_k) := \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k^2} + \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k \partial u_k} K + K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k \partial x_k} + K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k^2} K.$$

Since f is twice differentiable continuous, $C_M(\cdot)$ is continuous in the set Ω_0. For simplicity, denote

$$C_{\max} := \sup_{\xi_k \in \Omega_0} \|C_M(\xi_k)\|.$$

Thus, $\|\phi_k\| \le \frac{1}{2} C_{\max} \|x_k\|^2$ for all $x_k \in \Omega_0$. Furthermore,

$$\begin{aligned}
&\phi_k^T P \phi_k + x_k^T A_k^T P \phi_k + \phi_k^T P A_k x_k \\
\le&\frac{1}{4} C_{\max}^2 \|P\| \|x_k\|^4 + C_{\max} \|A_k\| \|P\| \|x_k\|^3.
\end{aligned} \tag{13}$$

Choose an $\alpha \in (0, \alpha_0]$ such that

$$\frac{\alpha C_{\max}^2 \|P\|}{4\lambda_{\min}(P)} + \sqrt{\frac{\alpha C_{\max}^2 \|A_k\|^2 \|P\|^2}{\lambda_{\min}(P)}} \le (\kappa^2 - 1) \lambda_{\min}(A_k^T P A_k)$$

for all $x_k \in \Omega$. Since $x_k^T x_k \le \frac{\alpha}{\lambda_{\min}(P)}$ for all $x_k \in \Omega$,

$$\begin{aligned}
&\phi_k^T P \phi_k + x_k^T A_k^T P \phi_k + \phi_k^T P A_k x_k \\
\le&\left(\frac{\alpha C_{\max}^2 \|P\|}{4\lambda_{\min}(P)} + \sqrt{\frac{\alpha C_{\max}^2 \|A_k\|^2 \|P\|^2}{\lambda_{\min}(P)}}\right) \|x_k\|^2 \\
\le&(\kappa^2 - 1) \lambda_{\min}(A_k^T P A_k) \|x_k\|^2.
\end{aligned} \tag{14}$$

Since $x_k^T A_k^T P A_k x_k \ge \lambda_{\min}(A_k^T P A_k) \|x_k\|^2$, one has

$$\phi_k^T P \phi_k + x_k^T A_k^T P \phi_k + \phi_k^T P A_k x_k \le (\kappa^2 - 1) x_k^T A_k^T P A_k x_k. \tag{15}$$

Substituting (15) into (12) yields that

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k \le \kappa^2 x_k^T A_k^2 P A_k x_k - x_k^T P x_k. \tag{16}$$

Using the Lyapunov equation (10), one has then

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k \le -x_k^T (Q + K^T R K) x_k. \tag{17}$$

Since $P > 0$, $Q + K^T R K > 0$ and A_k is asymptotically stable, inequality (17) implies that the set Ω is invariant for the nonlinear system (3) with the linear control law $u = Kx$.

Then, for any $x \in \Omega$, adding up (17) from 0 to ∞ with the initial condition $x_0 = x$ yields the desired results (11). □

Remark 2.1: Lemma 2 shows the existence of the set Ω where the nonlinear system (3) with the linear control law $u = Kx$ satisfies the considered constraints, and is invariant.

The linear control law Kx, also known as the terminal control law, is not applied directly to the plant, but just employed to calculate the terminal cost $\|x_{k+N|k}\|_P^2$ and terminal region Ω. The terminal cost is a local control Lyapunov function and satisfies inequality (10) in the terminal region. The terminal region, a level set of the terminal cost function, is positively invariant and renders all time-domain constraints to be satisfied.

III. Properties of the optimal cost function

Suppose that the optimization problem has an optimal solution for $x \in \mathcal{X}$, and define the related optimal cost function as

$$E(x) := \min_{U_k} J\left(x, U_k\right).$$

The optimal cost functional has the following properties.

Theorem 1: Considering the discrete-time nonlinear systems, the optimal cost function $E(x)$ has the properties as follows:

(i) $E(0) = 0$ and $E(x) > 0$ for all $x \neq 0$,

(ii) $E(x)$ is continuous at $x = 0$,

(iii) $E(x)$ is monotonically decreasing along the prediction trajectory, and

$$E(x_{k+1}) - E(x_k) \leq -\|x_k\|_Q^2 - \|u_k\|_R^2. \tag{18}$$

Proof: (i) Since $Q > 0$ and $R > 0$, $E(x) > 0$ as $x \neq 0$. Given $x_k = 0$, the optimal solution is $u^*_{k+i|k} \equiv 0$, and the corresponding optimal prediction trajectory is $x^*_{k+i+1|k} \equiv 0$, for all $i \in [0, N-1]$. Thus, $E(x_k) = 0$ as $x_k = 0$.

In terms of $R > 0$, we have the following inequality

$$E(x_k) \geq \sum_{i=0}^{N-1} \|x^*_{k+i|k}\|_Q^2 + \|x^*_{k+N|k}\|_P^2.$$

Obviously, only if $x^*_{k+i|k} \equiv 0$ for all $i \in [0, N]$, $E(x_k) = 0$ can be achieved. Therefore, for all $x_k \neq 0$, $E(x_k) > 0$.

(ii) In order to show the continuity of $E(x)$ at the equilibrium $x = 0$, pick up a point $x_k \in \Omega$ with $x_k \neq 0$. Then,

$$\bar{U}_k = \{Kx_{k|k}, Kx_{k+1|k}, \ldots, Kx_{k+N-1|k}\} \tag{19}$$

is a feasible solution to the optimization problem. In terms of Lemma 2,

$$\bar{E}(x_k) := x_k^T P x_k$$

is an upper bound of the finite horizon cost function. That is, $E(x_k) \leq \bar{E}(x_k)$ for all $x_k \in \Omega$.

Since $\bar{E}(x)$ is twice continuously differentiable, $\bar{E}(x)$ is continuous at $x = 0$. For any $\epsilon > 0$, there exists $\delta_0 > 0$ such that $|\bar{E}(x) - \bar{E}(0)| \leq \epsilon$ as $\|x - 0\| \leq \delta_0$.

Define a set

$$B_0 := \left\{x \in \mathbb{R}^{n_x} \mid x^T x \leq \frac{\alpha}{\lambda_{\max}(P)}\right\}.$$

Obviously, $B_0 \subseteq \Omega$. Denote $\delta := \min\left\{\delta_0, \sqrt{\frac{\alpha}{\lambda_{\max}(P)}}\right\}$, then $|E(x) - E(0)| \leq \epsilon$ for all $\|x - 0\| \leq \delta$. Therefore, the optimal cost function $E(x)$ is continuous at $x = 0$.

(iii) Suppose that at the time instant k the optimization problem has an optimal solution U_k^* which satisfies the control constraints (4b).

The related predicted state sequence is X_k^* which satisfies state constraints (4a), and $x^*_{k+N|k} \in \Omega$. The control sequence (8) guarantees the cost function

$$E(x_k) = \sum_{i=0}^{N-1} \|x^*_{k+i|k}\|_Q^2 + \|u^*_{k+i|k}\|_R^2 + \|x^*_{k+N|k}\|_P^2$$

is bounded. Implement the control $u_k = u^*_{k|k}$ into the systems (3). Since neither model perturbations nor external exogenous are considered, the system state at the time instant $k+1$ is

$$x_{k+1} = f(x_k, u^*_{k|k}),$$

which is the same as $x^*_{k+1|k}$. Thus, at the time instant $k+1$, choose the following control sequence

$$\begin{aligned} U_{k+1} &\triangleq \left[u_{k+1|k+1}, \quad \ldots, \quad u_{k+N-1|k+1}, \quad u_{k+N|k+1}\right] \\ &= \left[u^*_{k+1|k}, \quad \ldots, \quad u^*_{k+N-1|k}, \quad Kx^*_{k+N|k},\right], \end{aligned} \tag{20}$$

as a feasible solution to the optimization problem. The sequence U_{k+1} is composed of the shifted control sequence of U_K^*, starting from the time instant $k+1$, followed by the linear control law. Since $x^*_{k+N|k} \in \Omega$, $Kx^*_{k+N|k} \in \mathcal{U}$. Thus, the input constraint is satisfied for U_{k+1}.

In accordance with U_{k+1}, the state sequence is

$$\begin{aligned} x_{k+1+i|k+1} &= x^*_{k+1+i|k}, \qquad i \in [1, N-1], \\ x_{k+1+N|k+1} &= A_k x^*_{k+N|k}, \quad i = N. \end{aligned}$$

which satisfies the state constraints. As Ω is invariant under the linear control law, the terminal constraint is satisfied. The cost function at the time instant $k+1$ is

$$\begin{aligned} J_{k+1} =& \sum_{i=0}^{N-1} \left\|x_{k+1+i|k+1}\right\|_Q^2 + \left\|u_{k+1+i|k+1}\right\|_R^2 \\ &+ \left\|x_{k+1+N|k+1}\right\|_P^2 \\ =& \sum_{i=0}^{N-2} \left\|x^*_{k+1+i|k}\right\|_Q^2 + \left\|u^*_{k+1+i|k}\right\|_R^2 + \left\|x_{k+N|k+1}\right\|_Q^2 \\ &+ \left\|u_{k+N|k+1}\right\|_R^2 + \left\|x_{k+1+N|k+1}\right\|_P^2 \\ =& \sum_{i=0}^{N-1} \left\|x^*_{k+i|k}\right\|_Q^2 + \left\|u^*_{k+i|k}\right\|_R^2 - \left\|x^*_{k|k}\right\|_Q^2 \\ &- \left\|u^*_{k|k}\right\|_R^2 + \left\|x^*_{k+N|k}\right\|_Q^2 + \left\|Kx^*_{k+N|k}\right\|_R^2 \\ &+ \left\|A_k x^*_{k+N|k}\right\|_P^2 \\ =& E(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2 - \left\|x^*_{k+N|k}\right\|_P^2 \\ &+ \left\|x^*_{k+N|k}\right\|_{A_k^T P A_k}^2 + \left\|x^*_{k+N|k}\right\|_{Q+K^T RK}^2 \end{aligned}$$

Since $-P + A_k^T P A_k + Q + K^T R K \leq 0$,

$$J_{k+1} \leq E(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2 .$$

Since $E(x_k)$ is bounded, J_{k+1} is bounded. Thus, the candidate control sequence (20) is a feasible solution to Problem 1 at the time instant $k+1$. Furthermore, since the optimal solution is better than the feasible solution,

$$E(x_{k+1}) \leq J_{k+1} \leq E(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2 . \quad (21)$$

Therefore, $E(x)$ is monotonically decreasing along the system trajectory. □

Remark 3.1: In general, the optimal cost function is not necessarily continuous in x [11, 12, 14].

From the deduction of Theorem 1, it is easy to see that if the optimization problem has an optimal solution at the time instant k, then the optimization problem has a feasible solution at the time instant $k+1$. Thus, we come to the following conclusion:

Corollary 1: Suppose that Problem 1 has a feasible solution at the time instant $k=0$, Problem 1 has a feasible solution at each time instant $k>0$.

Proof: It is omitted. □

IV. Properties of systems under control

In this section, properties of systems under model predictive control are discussed. Convergence to the equilibrium is proven by the monotonically decreasing of the optimal cost function along the predicted trajectory, and Lyapunov stability is proven by a candidate Lyapunov function which is locally continuous. Asymptotic stability is proven in terms of convergence together with Lyapunov stability locally. In the end, inherent robustness locally in the terminal set is proven directly.

Lemma 3: Suppose that

(i) Assumptions 1-3 are satisfied

(ii) At the time instant $k=0$, Problem 1 has a feasible solution,

then,

(1) x_k converges to the set Ω in finite time,

(2) $\lim\limits_{k\to\infty} x_k = 0$.

Proof: (1) Firstly, it is proven by reduction to absurdity that there exists a finitely positive constant N such that $x_k \in B_0$ when $k > N$. Let $\Phi(i;x)$ be the solution of the system under control that starts from initial state x at time i. Suppose contrary to what is to be proven that $x_k \notin B_0$ for any $k>0$. Then by iterating inequality (18), we obtain

$$E\left(\Phi(k;x)\right) \leq -\sum_{i=1}^{k-1} \|\Phi(i;x)\|_Q^2 + E(x) .$$

Since $\|\Phi(k;x)\| > \sqrt{\frac{\alpha}{\lambda_{\max}(P)}}$, for any $i>0$,

$$E\left(\Phi(k;x)\right) < -\sum_{i=1}^{k-1} \frac{\alpha\lambda_{\min}(Q)}{\lambda_{\max}(P)} + E(x) .$$

Hence we have $E\left(\Phi(k;x)\right) \stackrel{k\to\infty}{\longrightarrow} -\infty$, which contradicts $E(x) \geq 0$ obviously.

(2) According to inequality (18), $E(x)$ is monotonically non-increasing. Furthermore, the lower bound of $E(x)$ is $E(0)=0$. Thus, $E(x)$ is convergent as $k\to\infty$ for bounded and monotonic sequence has a finite limit. By taking limits on both sides of inequality (18), we have

$$\lim_{k\to\infty} \|x_k\|_Q^2 \leq \lim_{k\to\infty} E(x_k) - \lim_{k\to\infty} E(x_{k+1}) = 0 .$$

Thus, $x_k \to 0$ as $t \to \infty$. □

In principle, the concept of stability only reflects a local property of the system at the equilibrium. Thus, in the following

$$V(x) := x^T P x, \quad \forall x \in \Omega,$$

is chosen as a candidate Lyapunov function which is an upper bound of the optimal cost function $E(x)$, and continuous differentiable in x.

Theorem 2: Suppose that

(i) Assumptions 1-3 are satisfied

(ii) At the time instant $k=0$, Problem 1 has a feasible solution,

then, the closed-loop system is asymptotically stable.

Proof: (1) Since $V(0)=0$, $V(x)>0$ for all $x \in \Omega$ with $x \neq 0$, and

$$V(x_{k+1}) - V(x_k) \leq -x_k^T \left(Q + K^T R K\right) x_k,$$

cf. Equ.(10), the system under finite horizon model predictive control is stable in terms of Lemma 1.

(2) Since x_k converges to Ω in finite time if Problem 1 has a feasible solution at the time instant $k=0$, and $\lim_{k\to\infty} x_k = 0$, the system is asymptotically stable. □

Inherent robustness means robustness of the closed-loop system using model predictive control, i.e., some disturbance or perturbation can be ignored [1]. Even discontinuous closed loops can still enjoy robustness suppose that a continuous Lyapunov function exists [11, 15].

Lemma 4: [11, 15] If there exists a Lyapunov function S on the basin of attraction $\mathcal{F}$ that is continuous on the interior of $\mathcal{F}$, then the origin of the system under control is robustly asymptotically stable on the interior of $\mathcal{F}$.

The next theorem shows the inherent robustness of model predictive control on the set Ω, that is, the perturbed solution will convergence to a small vicinity of the equilibrium for small sized perturbations or disturbances.

Theorem 3: Suppose that

(i) Assumptions 1-3 are satisfied

(ii) At the time instant $k=0$, Problem 1 has a feasible solution,

then, the closed-loop system is robustly asymptotically stable on the interior of Ω.

Proof: Consider the Lyapuonv function $V(x)$ of system (3) on the set Ω. Since $V(x)$ is twice continuously differentiable in the system state x, the system under model predictive control is robustly asymptotically stable on the interior of Ω which is derived directly from Lemma 4. □

V. CONCLUSIONS

In this paper, model predictive control of discrete time nonlinear systems was revisited. Firstly, the existence of the terminal set and terminal penalty was proven. Then the properties of the optimal cost function were investigated: continuous at its equilibrium, monotonically decreasing along the system trajectory. Compared with the existing papers, here convergence and stability were proven separately, cf. the convergence was proven in terms of the monotonically decreasing of the optimal cost function and stability was proven as a twice continuously differentiable function was used as the candidate Lyapunov function. Thus, the system is asymptotically stable since the system state converges to the equilibrium, and the system is stable. Furthermore, it is robustly asymptotically stable locally since the twice continuously differentiable function was used as the candidate Lyapunov function.

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